UNIVERSITY OF SOUTHAMPTON

ABSTRACT

FACULTY OF ENGINEERING AND APPLIED SCIENCE

SHIP SCIENCE

Doctor of Philosophy

THE PROCESSING OF DATA FROM MULTI-HYDROPHONE TOWED ARRAYS OF UNCERTAIN SHAPE

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An array of omni-directional hydrophones in tow is used to locate distant sources of acoustic radiation.

Where it is impossible, either actually or virtually, either to rotate the antenna or to change its shape, it is expedient to maximize the parallactic angle of point source and antenna through lengthening the antenna, and, since the antenna in question is implemented in the form of a string of discrete elements, to maximize the noise-rejective potential of the antenna by maximizing the number of elements in the string.

Although *a priori* the placing of hydrophones in an array is influenced by an uncertainty in knowledge of array disposition, an uncertainty which increases with distance from the towing vessel, for convenience an actual array with hydrophones spaced equidistantly is assumed for most of the thesis, although a modicum of flexibility of the antenna is allowed. In practice, the appropriateness or otherwise of a particular disposition of hydrophones is a function of the actual location and spectral character of a source.

In virtue of the uncertainty of sensor location as well as of modest relative motion of source and array, phase-differences of signal, reflected by measured pressures compared between hydrophones, are surmised in terms of bands of tolerance. It is shown that three such phase 'bins' per wavelength is optimal in a novel method presented in the thesis for comparing and contrasting the contents of bins such that a maximum may be associated uniquely with the location of a source.

The thesis is submitted with the conviction that a practical solution to a contemporary given problem of 'fuzzy' instrumentation has been found, a solution elaborated upon a theoretical basis with which, taking account of modern facilities for practical implementation, advances in accuracy and speed of processing beyond existing limits may be achieved. The thesis is submitted in the hope that, by varying inductive and deductive patterns of reasoning, a contribution will have been made to the theoretical basis for eliciting unique solutions to fuzzy problems, for which a calculus as well as appropriate modes of algebraic and statistical logic may be requiring to be developed.

3 The Theoretical Basis of a Distinction Between Near-Field and Far-Field

3.1 The Data Model Assumed in the Thesis

In order to derive a differential equation, in which is reflected the propagation of acoustic disturbances in a fluid medium, we adopt a common approach and assume, *inter alia* and fundamentally, that physical quantities in fluid mechanics may be expressed as sums of state-steady values plus acoustic ones.[49] In the plane $z = 0$ the pressure $p(x, y, t)$ shall be given as follows:

$$p(x, y, t) = p_0(x, y) + p'(x, y, t), \tag{8}$$

where

$$p_0 = \frac{1}{T} \int_0^T p \, dt, \tag{9}$$

i. e. the ambient value in the location and circumstances of interest, and where the form p' should convey to the reader the sense of 'instantaneous value'.

p' is periodic with long-time mean value zero.

It seems that many assumptions are requiring to be made in order to derive a simple equation. In practice, factors are likely to be present and effective that may damage or invalidate any such simple wave solution.

Let $\underline{\phi}$ be the acoustic potential. Then

$$\nabla^2 \underline{\phi} = \frac{1}{c^2} \frac{\partial^2 \underline{\phi}}{\partial t^2}. \tag{10}$$

Take the origin at the centre of a small pulsating spherical surface radiating sound in all directions. Then the wave equation can be written as

$$\nabla^2 \underline{\phi} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial \underline{\phi}}{\partial r}) \tag{11}$$

$$= \frac{1}{r} \frac{\partial^2 (r \underline{\phi})}{\partial r^2}. \tag{12}$$

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[49]See D. Ross, *Mechanics of Underwater Noise*, New York, 1976, p. 23

Hence write

$$\underline{\phi} = \frac{f(r,t)}{r}. \tag{13}$$

Then

$$\frac{\partial^2 f}{\partial r^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}. \tag{14}$$

Equation 14 is the one-dimensional wave equation, with the radius r as the co-ordinate. The solution of Equation 14 has the form

$$f = f_1(ct - r) + f_2(ct + r), \tag{15}$$

where f_1 and f_2 are arbitrary functions. Thus the general solution of Equation 12 is of the form

$$\underline{\phi} = \frac{f_1(ct - r)}{r} + \frac{f_2(ct + r)}{r}. \tag{16}$$

The first term of Equation 16 is an out-going wave, while the second is a wave coming in to the origin. An out-going monochromatic spherical wave $\underline{\phi}$ is given by

$$\underline{\phi} = \frac{A}{r}\, e^{i\,(kr - \omega t)}, \tag{17}$$

and an in-coming one $\underline{\phi}$ is given by

$$\underline{\phi} = \frac{A}{r}\, e^{i\,(\omega t - kr)}. \tag{18}$$

Returning to the general three-dimensional arrangement, if $\underline{\phi}$ varies sinusoidally with time, *i. e.* if

$$\underline{\phi} = e^{i\omega t}\,\Phi(\underline{r}), \tag{19}$$

then

$$\frac{\partial^2 \underline{\phi}}{\partial t^2} = -\omega^2 e^{i\omega t}\,\Phi(\underline{r}), \tag{20}$$

and

$$\nabla^2 \underline{\phi} = e^{i\omega t}\,\nabla^2 \Phi(\underline{r}). \tag{21}$$

Hence the wave equation becomes

$$\nabla^2 \Phi = -(\frac{\omega}{c})^2 \Phi, \tag{22}$$

i. e.

$$\nabla^2\Phi + k^2\Phi = 0 \text{ (Helmholtz),}^{50} \tag{23}$$

where

$$k = \frac{\omega}{c} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda}. \tag{24}$$

Turning now to the acoustic field created by a point source, consider a small hollow spherical shell, with long-time mean radius a_0, of which the cavity volume undergoes a very small uniform periodic fluctuation. This fluctuation gives rise to a mass flux Q in the medium, as follows:

$$Q(t) = \rho_0 \dot{V}(t) = Q_0 \cos \omega t, \tag{25}$$

where ρ_0 is the density of the fluid at the temperature of interest and $\dot{V}(t)$ is the rate of change of volume. With small fluctuations in volume, Q may be expressed in terms of the area and the radial velocity of the surface of the shell, *i. e.*

$$Q(t) \approx \rho_0(4\pi a_0^2)\,\dot{a} = \rho_0 S_0 u, \tag{26}$$

where $\dot{a}$ is the radial velocity, S_0 is the long-time mean surface area of the shell, *i. e.*

$$S_0 = 4\pi a_0^2, \tag{27}$$

and where

$$\underline{u} = u_0 e^{i\omega t} = \frac{Q_0}{4\pi a_0^2 \rho_0} e^{i\omega t}. \tag{28}$$

Insofar as the vibrating surface is everywhere in contact with the fluid, the instantaneous particle speed $v'(a_0)$ must be equal to the surface vibratory speed u. Given such equality, the acoustic pressure at the surface of the shell may be calculated by multiplying u by the specific acoustic impedance $\underline{z}_a$, evaluated at the surface. Whereas it can be shown that the instantaneous particle velocity $\underline{v}'$ is given by

$$\underline{v}' = ik\underline{\phi}\,(1 - (\frac{i}{kr})), \tag{29}$$

and the instantaneous pressure $\underline{p}'$ by

$$\underline{p}' = i\rho_0\omega\underline{\phi}, \tag{30}$$

[50]See C. A. Coulson and A. Jeffrey, *Waves*, London, 1988 (first published 1941), p. 141

so the specific acoustic impedance $\underline{z}_a$, by definition, is given by

$$\underline{z}_a = \frac{\underline{p}'}{\underline{v}'} = \frac{\rho_0 \omega}{1 - (\frac{i}{kr})}. \tag{31}$$

Now, with c_0 understood as the long-time speed of sound for the fluid in question,

$$\underline{z}_a(a_0)\, \underline{v}'(a_0) = \underline{p}'(a_0) \tag{32}$$

$$\begin{aligned}
&= \rho_0 c_0 \frac{(ka_0)^2 + i(ka_0)}{1 + (ka_0)^2} \underline{u} \\
&= \rho_0 c_0 \frac{(ka_0)^2 + i(ka_0)}{1 + (ka_0)^2} \frac{Q_0}{4\pi a_0^2 \rho_0} e^{i\omega t} \\
&= \frac{Q_0 c_0}{4\pi} e^{i\omega t} \frac{k^2 + \frac{ik}{a_0}}{1 + (ka_0)^2} \\
&= \frac{Q_0 \omega}{4\pi(\frac{\omega}{c_0})} e^{i\omega t} \frac{k^2 + \frac{ik}{a_0}}{1 + (ka_0)^2} \\
&= \frac{\omega Q_0}{4\pi} e^{i\omega t} \frac{k + \frac{i}{a_0}}{1 + (ka_0)^2} \\
&= \frac{\omega Q_0}{4\pi a_0} e^{i\omega t} \frac{ka_0 + i}{1 + (ka_0)^2} \\
&= \frac{\omega Q_0}{4\pi a_0} \frac{1}{\sqrt{1 + (ka_0)^2}} e^{i(\omega t + \theta_a)},
\end{aligned} \tag{33}$$

where

$$\theta_a = \arctan(\frac{1}{ka_0}) \approx \frac{\pi}{2} - \arctan(ka_0), \tag{34}$$

if

$$a_0 \ll \frac{1}{k} = \frac{1}{(\frac{\omega}{c_0})} = \frac{c_0}{\omega}. \tag{35}$$

It follows that we can give the instantaneous pressure $\underline{p}'(r)$ at distance r from the centre of the sphere as follows:

$$\underline{p}'(r) = \frac{a_0}{r} \underline{p}'(a_0)\, e^{-ik(r - a_0)} \tag{36}$$

$$= \frac{a_0}{r}\frac{\omega Q_0}{4\pi a_0}\frac{1}{\sqrt{1 + (ka_0)^2}} e^{i(\omega t + \theta_a)} e^{-ik(r - a_0)} \tag{37}$$

$$= \frac{\omega Q_0}{4\pi r}\frac{e^{i(\theta_a + ka_0)}}{\sqrt{1 + (ka_0)^2}} e^{i(\omega t - kr)}. \tag{38}$$

If

$$a_0 \ll \frac{1}{2\pi k}, \tag{39}$$

where a_0, again, is the long-time average radius of the spherical shell, then

$$\theta_a \approx \frac{\pi}{2} - ka_0, \tag{40}$$

and we can take the second term of Equation 38 as i, *i. e.*

$$\frac{e^{i(\theta_a + ka_0)}}{\sqrt{1 + (ka_0)^2}} \approx i. \tag{41}$$

In summary, then, if the mean radius a_0 of the radiating hollow spherical shell is very small compared to an acoustic wavelength, then the instantaneous radiated pressure $\underline{p}'(r)$ at a distance r from the centre of the source is given by

$$\underline{p}'(r) = i\,\frac{\omega Q_0}{4\pi r}\, e^{i(\omega t - kr)}. \tag{42}$$

6.2 Phase Binning

In a final-year undergraduate project the present author had developed a method of 'optimal binning'.[57] The problem was to elicit the periodicities of *Sigma Scorpii*, a close binary star, from noisy data. In particular, the measurements of the radial velocity of the star-system had been taken over a long period of time, of the order of decades indeed, and at wildly irregular known intervals.

It was found that the irregularity of the sampling intervals was responsible for the presence of aliases of the true periods on the periodogram obtained from applying the Fourier Transform to the data. The present author conducted a simple experiment with artificial data. First, he applied the Fourier Transform to samples taken at regular intervals. The periods of the artificial data appeared as expected with only their expected integer multiples as aliases. Next, he applied the Fourier Transform to the artificial data at the same irregular intervals as those at which the radial velocities of *Sigma Scorpii* had been measured. This time, however, not only did the true periods and their integer-multiple aliases appear, but also a number of aliases associated with other, spurious periods occurring on the periodogram.

Thus the present author had had some experience of dealing with irregular sampling intervals before embarking upon the present research. However, the difference was that, with *Sigma Scorpii* the irregularity was known, whereas with the towed array it was not.

In his dissertation upon *Sigma Scorpii* the present author developed a rapid method for estimating periods where the sampling was irregular. For an arbitrary period, the entire sample-series was 'folded', each sample being assigned a phase 'bin' rather than a phase point. A bin is a band of phase points. When every 'folded' sample had been assigned its appropriate bin, the contents of each bin were averaged and squared then added together. If a period of the radial velocity had been found, the sum was large. But if the test period was not a period of the source, the contents of the bins tended to cancel each other out, and the sum was negligibly small.

The author found that the optimal number of bins was three, and he will justify this assertion with a theorem and proof below.

The phase-binning method is quicker than a discrete Fourier Transform

[57] See G. W. Sweet, *The Periodicities of* Sigma Scorpii, Dissertation, Oxford Polytechnic, 1990

because it does not involve convolution with sines or cosines. Instead, it employs only the 'mod' operation.

The algorithm, when applied to the problem of the point source and the towed array, proceeds as follows. The wavelength of the source is divided into equal parts called 'phase bins'. The choice of bin in which a pressure, measured by means of a hydrophone, is placed is determined by the supposed frequency and location of the source of interest. Naively, if the binning is right, the measured pressures will reinforce each other. But if measured pressures are assigned to the wrong bins, they will tend to cancel each other out.

A bin represents a range of phase points. Because of the uncertainty of hydrophone location, a measured pressure should be thought of as containing potential phase angle information about a range between points on the real line rather than about a unique point. In other words, we regard data coming from hydrophones as being capable of yielding 'fuzzy' information at best.

For a single snapshot, in the absence of noise, there must be at least three bins per wavelength. Three is the minimum number of bins required to corroborate a wavelength without regard to the phase angle of the signal. If d is the distance between hydrophones reached consecutively by a wavefront, the condition

$$d < \frac{\lambda}{2} \tag{176}$$

applies.

In general, while the period or location of the source of a signal may be conceived of in terms of non-rational numbers, they cannot in practice be measured or characterized except in terms of rational ones. In this sense, an analysis based upon the convolution of a measured signal with a notional signal is limited. The method of phase binning requires the employment of notional periods and locations but not the convolution with a notional signal. Indeed, the convolution with a notional signal may in any case be limited by the uncertainty of hydrophone location, especially if the convolution was of such a kind that phase angle was requiring to be matched with phase angle, *i. e.* point with point. Instead, we propose a sort of 'convolution' of a band with band of points. Indeed, if the parallactic angle of source and array is sufficiently wide to allow a good sense of the distance of the source, we believe it possible to think of our 'convolution' in terms of region with region. In other words, we are trying to stretch the notion of convolution from point to line and then to area, and thus expand upon the dimensional potential of convolving.

In presenting now the algebraic elements of our method, we hope not to do violence to our policy of specializing in the location of point sources near enough to be locatable in any event if we treat plane waves in what follows. This is done for passing convenience. We shall go on to consider the general case of palpably spherical spreading subsequently, for which a more differentiated approach will be required.

6.3 The Ratio P/a^2

In the absence of noise and in a medium of infinite extent let p be measured pressures with

$$p = a \sin\theta \ . \tag{177}$$

Let B_n ($n = 1, 2, 3$) denote phase bins. Then

$$B_1 = (\overline{p})_1 = \frac{1}{2\pi/3}\int_0^{\frac{2\pi}{3}} a \sin\theta d\theta = \frac{9a}{4\pi} \tag{178}$$

$$B_2 = (\overline{p})_2 = \frac{1}{2\pi/3}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} a \sin\theta d\theta = 0 \tag{179}$$

$$B_3 = (\overline{p})_3 = \frac{1}{2\pi/3}\int_{\frac{4\pi}{3}}^{2\pi} a \sin\theta d\theta = -\frac{9a}{4\pi} \ . \tag{180}$$

Construct the power integral P as follows:

$$P = (B_1)^2 + (B_2)^2 + (B_3)^2 = \frac{81a^2}{8\pi^2} = 1.0259a^2. \tag{181}$$

Therefore

$$\frac{P}{a^2} = 1.0259 \ . \tag{182}$$

It will be shown below that, in general, with N bins

$$\frac{P}{a^2} = \frac{N^3}{2\pi^2}\sin^2(\frac{\pi}{N}) \tag{183}$$

in the limit either as $\lambda \to \infty$ with length of array l (assuming, for convenience, a straight antenna of known orientation) and the number of hydrophones M constant or, with a given λ and l, as $M \to \infty$.

To treat noise as effectively as possible, we must make sure that $\frac{P}{a^2}$ is kept as small as possible. This is achieved when $N = 3$, which is the smallest possible value for N.

6.4 The Merits of an 'Exponential' Spacing of Hydrophones

The two Figures following have been included to point up what we believe to be an advantage of spacing hydrophones non-equidistantly along a straight array. Suppose the antenna were straight, and that a far distant point source was radiating upon the antenna along end-fire. Suppose, further, that the wavelength of the source was $2d$. Then all items in each bin would have the same value, and no integration could be undertaken over the period of the signal in the sense of the Method outlined above.

We believe that singularities arising for that and such reasons with an array of equidistantly-spaced hydrophones may be treated by spacing the hydrophones non-equidistantly. The graph in Figure 12 shows the plot of a curve which is not smooth. But it has become much smoother in Figure 13, where the coordinates $(x_m, 0)$ $(m = 1, 2, ...)$ of the hydrophones h_m are given as follows:

$$x_m = 1000(e^{\frac{m-1}{1000}} - 1) \ . \tag{184}$$

The thus 'exponential' spacing would seem to allow a better spread of values across a bin. As we shall argue shortly, a more or less good spread of values across bins becomes possible with greater gross flexing of the array.

With Figures 12 and 13 there is no noise on the signal. but there is an uncertainty of one part in a thousand in the coordinates of the hydrophones.

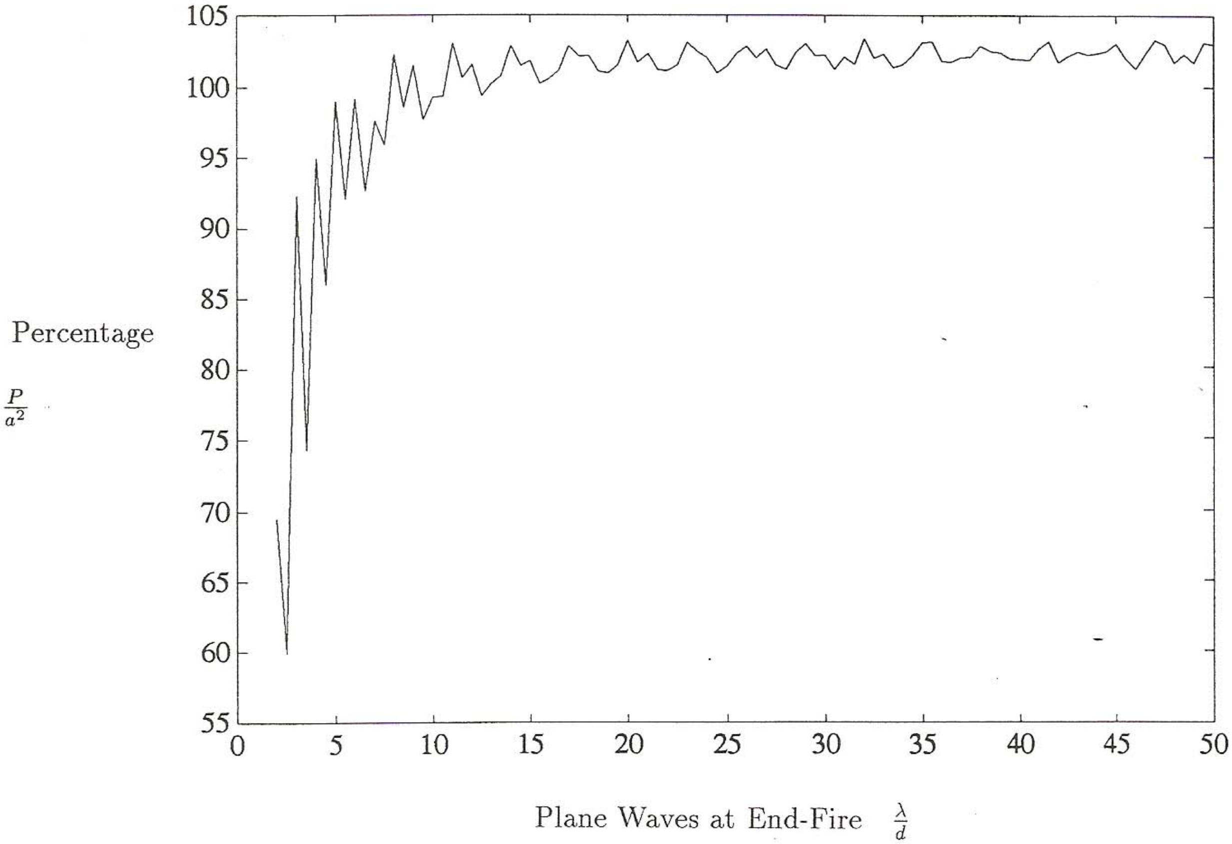

Figure 12: Singularities with even spacing of hydrophones

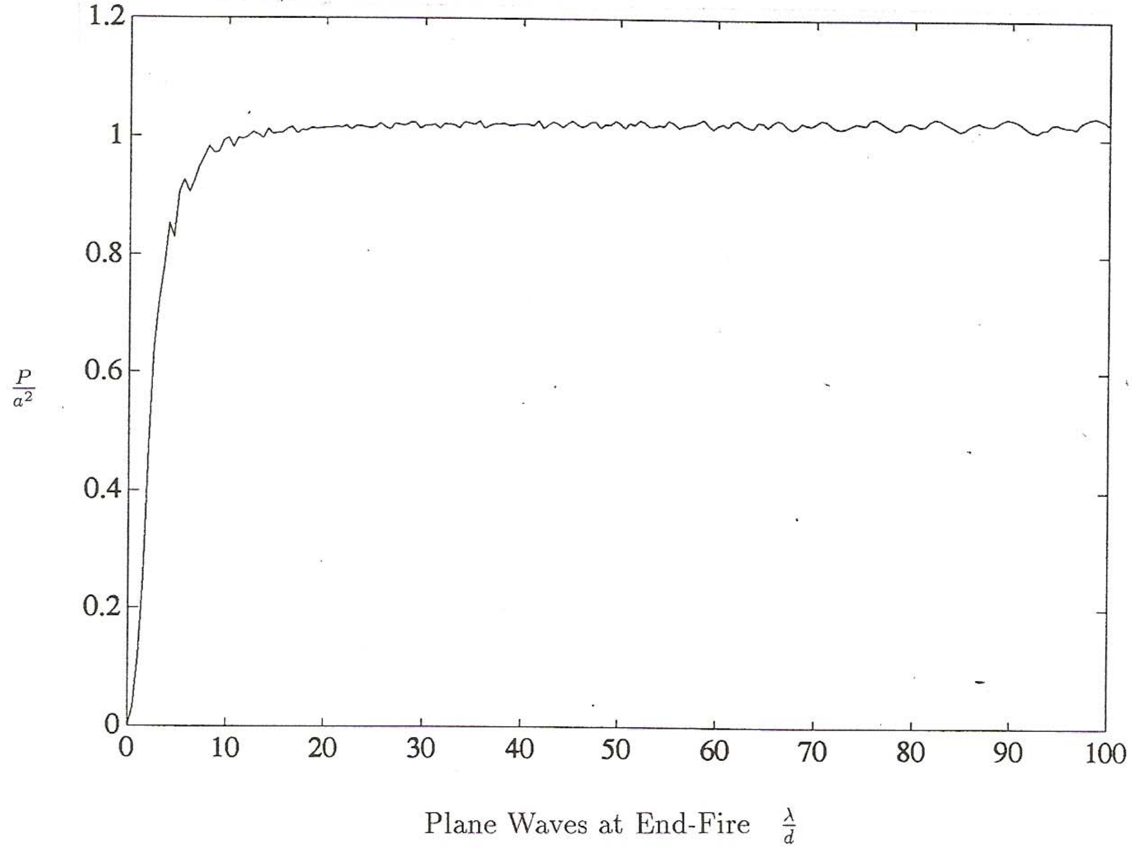

Figure 13: Treatment of singularities with 'exponential' spacing of hydrophones

6.5 Determination of Amplitude of Plane Wave

For simplicity, let the array be assumed to be straight, the hydrophones spaced equidistantly, and the source sufficiently far away for the wave front to be planar upon arrival at the array. Let L be the length of the array, M the number of hydrophones, λ_0 the wavelength of the plane wave, a the amplitude of the wave, θ the angle of incidence of the wave upon the antenna, N the number of bins and P the power in the binning periodogram at $\lambda = \lambda_0 \sec\theta$.

6.5.1 Theorem

In the limit as $M \to \infty$, and for values of λ_0 for which $L\cos\theta/\lambda_0$ is an integer K,

$$\frac{P}{a^2} = \frac{N^3}{2\pi^2}\sin^2(\frac{\pi}{N}) \ . \tag{185}$$

E. g.

$$\begin{aligned} \frac{P}{a^2} &= 1.0259, \quad N = 3 \\ &= 1.6211, \quad N = 4 \\ &= 2.1879, \quad N = 5 \\ &\sim \frac{1}{2}N, \quad \text{large } N. \end{aligned} \tag{186}$$

Thus $\frac{P}{a^2}$ is least when $N = 3$. It can be shown that this makes the estimate of a^2 least sensitive to noise on the signal. Hence the term 'optimal binning' when $N = 3$.

6.5.2 Proof

$$p(\underline{r}, t) = a\cos(2\pi\nu_0 t - 2\pi\underline{r}.\underline{l}/\lambda_0 + \phi_0) \ , \tag{187}$$

where $\underline{l} = (\cos\theta, -\sin\theta)$.

Now $\underline{r} = (x, 0)$ at point x on the array, therefore at time $t = t_0$

$$p(x) = a\cos(\frac{2\pi\nu}{\lambda} + \phi) = a\cos(\chi + \phi) =, \quad \lambda = \lambda_0\sec\theta, \tag{188}$$

where

$$\begin{aligned} \chi &= \frac{2\pi x}{\lambda}, \\ \phi &= -2\pi\nu_0 t_0 - \phi_0. \end{aligned} \tag{189}$$

In binning procedure (in limit $M \to \infty$)

$$\begin{aligned} B_1 = (\overline{p})_1 = \frac{N}{2\pi}\int_0^{\frac{2\pi}{N}} a\cos(\chi+\phi)\,d\chi &= \frac{Na}{2\pi}\left[\sin(\chi+\phi)\right]_0^{\frac{2\pi}{N}} \\ &= \frac{Na}{2\pi}\left(\sin(\frac{2\pi}{N}+\phi) - \sin\phi\right). \end{aligned}$$

Therefore

$$B_1 = \frac{Na}{\pi}\sin(\frac{\pi}{N})\cos(\phi+\frac{\pi}{N}). \tag{190}$$

Similarly

$$\begin{aligned} B_2 = (\overline{p})_2 = \frac{N}{2\pi}\int_{\frac{2\pi}{N}}^{\frac{4\pi}{N}} a\cos(\chi+\phi)\,d\chi &= \frac{Na}{\pi}\sin(\frac{\pi}{N})\cos(\phi+\frac{3\pi}{N}) \\ &\vdots \\ &\vdots \\ B_n = (\overline{p})_n = \frac{N}{2\pi}\int_{\frac{2\pi(n-1)}{N}}^{\frac{2\pi n}{N}} a\cos(\chi+\phi)\,d\chi &= \frac{Na}{\pi}\sin(\frac{\pi}{N})\cos\left(\phi+\frac{(2n-1)\pi}{N}\right) \\ &\vdots \\ &\vdots \\ B_N = (\overline{p})_N = \frac{N}{2\pi}\int_{\frac{2\pi(N-1)}{N}}^{2\pi} a\cos(\chi+\phi)\,d\chi &= \frac{Na}{\pi}\sin(\frac{\pi}{N})\cos\left(\phi+\frac{(2N-1)\pi}{N}\right) \end{aligned}$$

$\Rightarrow$

$$P = B_1^2 + B_2^2 + \ldots + B_N^2 = a^2(\frac{N}{\pi})^2\sin^2(\frac{\pi}{N})\sum_{n=1}^{N}\cos^2\left(\phi+\frac{(2n-1)\pi}{N}\right) = \frac{1}{2}N \tag{191}$$

from Lemma below (all integer $N > 2$), therefore

$$\frac{P}{a^2} = \frac{N^3}{2\pi^2}\sin^2(\frac{\pi}{N}), \quad \textit{quod erat demonstrandum.} \tag{192}$$

6.5.3 Lemma

$$\sum_{n=1}^{N} \cos^2\left(\phi + \frac{(2n-1)\pi}{N}\right) = \frac{1}{2}N, \text{ all integer } N > 2. \tag{193}$$

6.5.4 Proof

$$\begin{aligned}
I_N = \sum_{n=1}^{N} \cos^2\left(\phi + \frac{(2n-1)\pi}{N}\right) &= \frac{1}{2}\sum_{n=0}^{N-1}\left(1 - \cos\left(\frac{2\phi + 2\pi(2n+1)}{N}\right)\right) \\
&= \frac{1}{2}N - \frac{1}{2}\Re\sum_{n=0}^{N-1} e^{2i\phi + \frac{2\pi i}{N}}.e^{\frac{4\pi i n}{N}} \\
&= \frac{1}{2}N - \frac{1}{2}\Re e^{2i\phi + \frac{2\pi i}{N}} \underbrace{\sum_{n=0}^{N-1} e^{\frac{4\pi i n}{N}}}_{} \\
&\qquad\qquad = \frac{e^{\frac{4\pi i N}{N}} - 1}{e^{\frac{4\pi i}{N}} - 1} \\
&\qquad\qquad = 0, \text{ all integer } N > 2
\end{aligned}$$

$\Rightarrow$

$$I_N = \frac{1}{2}N, \text{ integer } N > 2, \textit{ quod erat demonstrandum.} \tag{194}$$

N. B. When $L\cos\theta/\lambda_0 \neq$ integer, then $B_n \neq (\overline{p})_n$ exactly in the binning procedure; for, after taking the mean of the sums over the $K = [L/\lambda]$ wavelengths λ (where $\lambda = \lambda_0 \sec\theta$) along the array, there is a quantity of order K left over at the far end.